Interval Sequences In 12-Tone Rows

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Abstract

The 12-tone rows in the music of Schoenberg, Webern, and Berg have been thoroughly analysed with respect to four transformations: transposition, retrograde, inversion, and a cyclic shift. The corresponding interval sequences, however, have received far less attention. We use a brute-force numerical search to map out some properties of the interval sequences in 12-tone rows.

1 Introduction

In the 1920s Arnold Schoenberg came up with the concept of a *12-tone row* (TTR) as a device for composing classical music, and it was later elaborated on by Webern and Berg [4]. A TTR is effectively a permutation of the 12 notes within an octave, and so there are 12!, or close to 480 million, such tone rows. As a way to add variation to a given TTR it is often modified by one or more of four transformations: transposition (shifting the pitches up or down by a constant amount), retrograde (reversing the sequence so that it plays backwards), inversion (flipping the direction of the pitch changes), and a cyclic shift (splitting the sequence up into two parts, and swapping them) [5, 8]. Consequently, most analysis has concentrated on how the set of TTRs is mapped onto itself by a combination of those four transformations. Since the set is very large not much progress was made until electronic computers became available in the 1960s [1, 2]. Nowadays, with the availability of sophisticated software for symbolic mathematics [9] some types of symmetry can be calculated with a few lines of code within seconds [8]. The set of *interval sequences* (IS), however, is much harder to analyse and have not received the same attention with the notable exception of All-Interval Sequences [1, 3], which are mentioned in Section 4.2.

In modern music, jazz in particular, intervallic soloing is occasionally taught. Bergonzi [6] uses a system where he systematically defines all combinations of 3, 4, or 5 intervals. The direction of the movement is indicated by a plus or minus sign (plus means the next pitch is higher, minus it is lower). He then selects a few examples from each group and presents them in musical notation.

In the following we consider the twelve notes to be represented by the non-negative integers from 0 to 11. We calculate the 1st-order difference modulo 12 of all possible permutations of the integers from 0 to 11, and list a few examples of their properties.

2 Interval Sequences

2.1 Ordered and Unordered Interval Sequences

As is standard in the field we consider musical notes with the same name to be identical even when they are in different octaves so that each note can be assigned an integer value in the range from 0 to 11 [1]. Similarly, when arithmetic is performed modulo 12 the interval in semitones between adjacent notes can be assigned a value also in the range from 0 to 11. As an example consider the theme from Schoenberg's Opus 25, Suite For Piano [11]. As a permutation $\mathbf{p}$ in vector form it can be written[1]

$$\mathbf{p} = \begin{bmatrix} 0 & 1 & 3 & 9 & 2 & 11 & 4 & 10 & 7 & 8 & 5 & 6 \end{bmatrix}. \tag{1}$$

The *ordered interval sequence* (oIS) $\mathbf{i}_o$ is calculated as the first order difference Δ modulo 12 of $\mathbf{p}$ where it is assumed that $\mathbf{p}$ is periodic (its first element is appended to the end) so that $\mathbf{i}_o$ also has 12 elements. Consequently, $\mathbf{i}_o$ corresponding to $\mathbf{p}$ from 1 is

$$\mathbf{i}_o = \Delta(\mathbf{p}) = \begin{bmatrix} 1 & 2 & 6 & 5 & 9 & 5 & 6 & 9 & 1 & 9 & 1 & 6 \end{bmatrix}. \tag{2}$$

The *unordered interval sequence* (uIS) $\mathbf{i}_u$ is $\mathbf{i}_o$ arranged into a sorted list, having its smallest element at the start. Consequently $\mathbf{i}_u$ corresponding to $\mathbf{p}$ from Equation 1 is

$$\mathbf{i}_u = \begin{bmatrix} 1 & 1 & 1 & 2 & 5 & 5 & 6 & 6 & 6 & 9 & 9 & 9 \end{bmatrix} \tag{3}$$

Indexing of the elements is assumed to start at 1, not 0. We note that when $\mathbf{i}_o$ and $\mathbf{i}_u$ are derived from a TTR, they can never contain the value 0. On the other hand, as opposed to a TTR, both $\mathbf{i}_o$ and $\mathbf{i}_u$ can contain repeated elements all the way up to the 12 identical elements that produce the chromatic scale (all 1s or all 11s) and the cycle of fifths (all 5s or all 7s).

2.2 Calculating Interval Sequences

Let us denote the set of all permutions of the integers from 0 to 11, which is equivalent to the set of TTRs, by P_{12}[2]. We can generate the complete set I_o of oIS by applying Δ to all the 12! elements of P_{12}. Furthermore, by sorting each $\mathbf{i}_o$ we can calculate I_u, the complete set of uIS. However, since transposing a TTR does not affect its oIS, just as the derivative of a function does not change when a constant is added, the number of unique elements in I_o is at most 11! which is 39,916,800, or just under 40 million. We therefore fix the starting note by setting the first element of $\mathbf{p}$ to 0.

The number crunching is done, brute-force, in Octave [10] running on a Linux PC. Permutations are generated by using a port of the algorithm from the Standard Template Library in C++ [7]. Each vector is packed into a single 8-byte integer by effectively encoding the

[1]Incidentally, in musical notation the notes are E, F, G, Db, Gb, Eb, Ab, D, B, C, A, Bb. The starting note is often assigned a value of 0 regardless of which note it is.

[2]We deliberately do not use the symbol S_{12} because our range starts at 0, not 1.

12 elements into an integer base 16[3]. Calculating the 11! members of the set I_o takes of the order of ten hours but once the encoding is completed it is fast to operate directly on the integers when sorting and searching. Operations that require unpacking of the integers into vectors, on the other hand, are slow.

3 General Properties Of Interval Sequences

3.1 Ordered Interval Sequences

In order to get a feel for how the elements $\mathbf{i}_o$ of I_o are distributed let us take a look at the set they are derived from, P_{12}. The number of vectors in P_{12} is 12!, or 479,001,600, whereas the number of vectors whose elements are integers in the range from 0 to 11 is 12^{12}, or 8,916,100,448,256, which is almost 20,000 times larger than $|P_{12}|$. So the TTRs are very thinly distributed in the space of unconstrained tone rows. In I_o there are 11!, or 39,916,800, vectors whereas the number of vectors in the set whose elements are in the range from 1 to 11 is 11^{12}, or 3,138,428,376,721, which is almost 80,000 times larger than $|I_o|$. Consequently, I_o is approximately four times sparser in the space of interval sequences without zeros than TTRs are in the space of tone rows. If you pick a vector $\mathbf{p}$ of 12 random notes, chances are 1 to 20,000 that it belongs to P_{12} whereas if you pick a vector $\mathbf{i}_o$ of 12 random ordered intervals without zeros, chances are 1 to 80,000 that it belongs to I_o.

3.2 Unordered Interval Sequences

Clearly, a random guess on $\mathbf{i}_o$, without any attempt to constrain its values, is hopelessly optimistic. For a start, the elements of $\mathbf{i}_o$ have to sum up to a multiple of 12, otherwise it cannot correspond to a tone row that is periodic. A more ambitious question is therefore, how well are the unordered interval sequences $\mathbf{i}_u$ represented among the partitions of multiples of 12 with 12 elements? The results are summarised in Table 1.

It is obvious that some partitions are not feasible. For example, a partition that contains only even numbers cannot generate any of the odd integers in a TTR. Nevertheless, the coverage is remarkably good, with 49,031 unique unordered interval sequences out of a possible 53,934. If you choose a random partition, using only the constraint that it must have 12 elements that sum up to a multiple of 12, chances are better than 90% that its elements can be reshuffled into a sequence $\mathbf{i}_o$ that belongs to I_o. That is in stark contrast to how sparse the ordered interval sequences are among the possible permutations of the 49,031 feasible partitions that make up I_u. There are 260,662,566,960 such permutations[4] compared to the 11! ordered interval sequences in I_o. So if you perform a random permutation of the elements of $\mathbf{i}_u$ then chances are 1 to 6,530 that the result belongs to I_o even when you know $\mathbf{i}_u$ belongs to I_u.

[3]The base must be at least 12, of course, and 16 was chosen in the hope that using a factor of two would speed up execution at the hardware level. Unfortunately, it does not seem to make it any faster, probably because the processor is wired up to perform calculations in floating point regardless of the variable type

[4]Calculated with a script in GAP [9].

Table 1: Number of $\mathbf{i}_u$ compared to partitions of multiples of 12.

Sum	Partitions	Unique $\mathbf{i}_u$	Sample $\mathbf{i}_u$
12	1	1	1,1,1,1,1,1,1,1,1,1,1,1
24	75	36	1,1,1,1,1,1,1,1,1,2,2,11
36	1,012	798	1,1,1,1,1,1,1,1,3,3,11,11
48	5,079	4,507	1,1,1,1,1,1,1,4,4,11,11,11
60	12,470	11,470	1,1,1,1,1,1,5,5,11,11,11,11
72	16,660	15,407	1,1,1,1,1,6,6,11,11,11,11,11
84	12,470	11,470	1,1,1,1,7,7,11,11,11,11,11,11
96	5,079	4,507	1,1,1,8,8,11,11,11,11,11,11,11
108	1,012	798	1,1,9,9,11,11,11,11,11,11,11,11
120	75	36	1,10,10,11,11,11,11,11,11,11,11,11
132	1	1	11,11,11,11,11,11,11,11,11,11,11,11
Total	53,934	49,031	

4 Selected Classes Of Interval Sequences

4.1 Period of $\mathbf{i}_o$ less than 12 under a cyclic shift

There are only 734 ordered interval sequences that repeat with a period of less than 12 under a cyclic shift[5]. Four of those are the well known chromatic scale, ascending (all 1s) and descending (all 11s), and the circle of fifths, ascending (all 7s) and descending (all 5s). Both repeat for any cyclic shift. Another four have a period of 2. They are 1,9 repeated 6 times; 3,7 repeated; 3,11 repeated (inverse of 1,9); and 5,9 repeated (inverse of 3,7). The results are summarised in Table 2.

Table 2: The five groups of $\mathbf{i}_o$ that repeat under a cyclic shift

Period	$\|I_o\|$/Period	Sample $\mathbf{i}_o$
1	4	1..., 5..., 7..., 11...
2	4	1,9..., 3,7..., 3,11..., 5,9...
3	20	1,1,7...
4	78	1,1,1,5...
6	628	1,1,1,1,7,7...

The cyclic sequences listed above correspond to the cosets highlighted by Hunter and Hippel [8] in Table 2. Their cosets of size 24 have a period of 1; size 48 a period of 2; size 72 a period of 3; size 96 a period of 4; and size 144 a period of 6.[6] Interestingly, $\mathbf{i}_o$ from different

[5] Not counting the cyclic shifts that do not correspond to the period. For example, if the sequence 1,1,7... is counted then the sequence 1,7,1... is implicitly also included

[6] The coset size is not compatible with our $|I_o|$/Period. The reference serves only to show that cyclic sequences have a high degree of symmetry.

groups can correspond to the same $\mathbf{i}_u$. For example, the sequence 1,1,7... with period 3 and the sequence 1,1,1,1,7,7... with period 6 both correspond to the $\mathbf{i}_u$ that contains 8 1s and 4 7s. There are 30 such $\mathbf{i}_u$.

4.2 Unique Intervals

Let us denote the number of unique values in an interval sequence by N_i. So for $\mathbf{i}_o$ in Equation 2 and $\mathbf{i}_u$ in Equation 3, N_i is 5. If we count the unique intervals in all the 11! interval sequences we get the results shown in Table 3.

Table 3: Number of $\mathbf{i}_u$ and $\mathbf{i}_u$ with $\mathbf{N}_i$ unique intervals.

N_i	$\lvert I_u \rvert$	$\lvert I_o \rvert$	$[\lvert I_o \rvert / \lvert I_u \rvert]$
1	4	4	1
2	24	140	6
3	324	8,472	26
4	2,423	157,056	65
5	10,880	1,629,912	150
6	17,251	7,050,672	409
7	12,826	13,962,864	1089
8	4,480	11,951,592	2,668
9	769	4,526,544	5,886
10	49	606,408	12,375
11	1	23,136	23,136
Total	49,031	39,916,800	814

The more unique intervals $\mathbf{i}_u$ contains, the more permutations can be created from it so it is not surprising that $|I_o|/|I_u|$ increases with N_i. Only one $\mathbf{i}_u$ contains all the integers from 1 to 11: the famous All-Interval Sequence (AIS). Traditionally, the AIS is assumed not to be periodic so the jump from the last note to the first, which has to be 6, is left out. In 1965 Bauer-Mengelberg and Ferentz [1] were the first to calculate the 1,928 AIS, using ingenious methods to reduce the number of candidates they had to include in an exhaustive search on a primitive computer (see also Morris and Starr [3]). Their list deliberately excludes inversions and implicitly assumes the last interval is 6. By excluding inversions they shorten their list by a factor of two, and by forcing the last interval to be 6 they capture only two out of the 12 variations created by a cyclic shift[7], thereby further shortening their list by a factor of six. Consequently, the number of ordered interval vectors is 2 times 6 times 1,928 which is 23,136.

[7]Since the interval 6 appears twice in each AIS, there are two ways it can be arranged so that its last element is 6

4.3 $\mathbf{i}_o$ Equals Its Retrograde

We consider interval sequences with left-right symmetry over both 11 and 12 elements. Symmetry over 11 elements ignores the last element, and it is effectively treating the sequence as if it is non-periodic. The two types of symmetry are referred to as R_{11} and R_{12}. Examples $\mathbf{p}_{11}$ and $\mathbf{p}_{12}$ of R_{11} and R_{12} interval sequences are

$$\mathbf{p_{11}} = \begin{bmatrix}2 & 4 & 3 & 2 & 5 & 11 & 5 & 2 & 3 & 4 & 2 & 5\end{bmatrix} \tag{4}$$

and

$$\mathbf{p_{12}} = \begin{bmatrix}9 & 1 & 7 & 6 & 9 & 10 & 10 & 9 & 6 & 7 & 1 & 9\end{bmatrix}. \tag{5}$$

R_{11} symmetry is mirrored across element 6, and the value of element 12 does not matter. R_{12} symmetry is mirrored across elements 6 and 7, which have to be identical, and element 12 must have the same value as element 1. Consequently, we would expect R_{11} to contain more interval sequences than R_{12}, and that is indeed the case. There are 23,040 interval sequences in R_{11} and 3,840 in R_{12}. Furthermore, in R_{11} sequences element 6 is always odd (and so is the ignored element 12) whereas in R_{12} sequences elements 6 and 7 can take any value except 6.

It is remarkable that when we remove from R_{12} the sequences that can be transformed onto themselves by a cyclic shift we are left with 1,928 sequences out of 3,840. We are also left with 1,928 sequences when we make the equivalent operation on the 23,136 All-Interval Sequences (remove the sequences that can be transformed onto themselves by a cyclic shift). It is tempting to think it is not a coincidence but hard to see a deeper connection between the two sets.

4.4 $\mathbf{i}_u$ Equals Its Inverse Retrograde

This property is equivalent to the histogram of $\mathbf{i}_u$ being symmetric around the value 6. A sample $\mathbf{i}_u$ is

$$\mathbf{i}_u = \begin{bmatrix}1 & 2 & 2 & 3 & 3 & 5 & 7 & 9 & 9 & 10 & 10 & 11\end{bmatrix} \tag{6}$$

There are 2160 $\mathbf{i}_o$ with this particular histogram and one of those is

$$\mathbf{i}_o = \begin{bmatrix}1 & 7 & 2 & 9 & 10 & 10 & 11 & 9 & 5 & 2 & 3 & 3\end{bmatrix}. \tag{7}$$

Such interval sequences are particularly interesting to musicians because the intervals from 6 to 11 can also be thought of as negative intervals from -1 to -6, and therefore a symmetric histogram indicates that for each positive interval there is a corresponding negative interval. In modulo-12 arithmetic we can then consider the sequences in Equations 6 and 7 as

$$\mathbf{i}_u = \begin{bmatrix}1 & 2 & 2 & 3 & 3 & 5 & -5 & -3 & -3 & -2 & -2 & -1\end{bmatrix} \quad \text{mod } 12 \tag{8}$$

and

$$\mathbf{i}_o = \begin{bmatrix}1 & -5 & 2 & -3 & -2 & -2 & -1 & -3 & 5 & 2 & 3 & 3\end{bmatrix} \quad \text{mod } 12. \tag{9}$$

where the elements now sum up to zero. There are 341 $\mathbf{i}_u$ whose histogram is symmetric around the value 6, and 186 of these contain the value 6 itself. There are 440,208 $\mathbf{i}_o$, or approximately 1% of all TTRs, associated with the 341 $\mathbf{i}_u$.

5 Conclusions

It has been known since 1965 there are 1,928 All-Interval Sequences (excluding inversions and 10 out of 12 of the possible cyclic shifts). With a brute-force approach we have determined some further properties of the interval sequences derived from TTRs. There are 49,031 unordered interval sequences, and the histograms of 341 of those are symmetric around the interval 6. There are only 734 ordered interval sequences that repeat with a period of less than 12; 3,840 that are left-right symmetric around elements 6 and 7; and 23,040 that are symmetric around element 6 (thus ignoring element 12).

Although an exhaustive search can be useful in a special case such as this, it does very little to help explain the underlying properties of interval sequences. The general problem is easy to define, though: given a positive integer N, what are the properties of the 1st-order difference modulo N applied to the cyclical permutation group of order N? Since the complexity grows factorially any numerical approach will quickly run into a brick wall, with N greater than 20 most likely out of reach forever. Clearly, an advance in theory is required.

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